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Spatiotemporal complexity of the cubic–quintic nonlinear Schrödinger equation

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Abstract. The integrability of the cubic–quintic nonlinear Schrödinger equation is investigated numerically. By analytically studying the linear stability of the homogeneous state and numerically solving such a continuum Hamiltonian dynamic system, we show that the quintic nonlinear term leads to a spatiotemporal complexity of wave fields, and illustrate that this behaviour is associated with the stochastic partition of energy in Fourier modes. In addition, we show the presence of the stochastic motion is due to the homoclinic orbit crossings.

1. Introduction

The cubic–quintic nonlinear Schrödinger equation (NSE)

$$i\partial_t E + \partial_{xx} E + |E|^2 E - g|E|^4 E = 0 \tag{1}$$

has recently received much attention due to its physical application, where g is a parameter. In nonlinear optics, the refractive index is expanded to the second order in electric field E, the well known cubic NSE is obtained. However, Aciolic *et al* [1] experimentally measured the optical susceptibilities of CdS_xSe_{1-x} doped glasses using a spatially resolved phase-matched multiwave mixing technique, and showed that fifth-order nonlinear effects should also be considered. In many-nucleon systems, usually, collision processes of heavy ion are described on classical or quasi-classical levels [2]; it is shown that in the quasi-classical limit the nuclear hydrodynamics equations with the Skyrme forces can be reduced to the non-relativistic cubic–quintic model for the corresponding choice of variable [3]. In Langmuir plasmas, the beat frequency interaction between the large amplitude parts of high-frequency Langmuir fields and ion-acoustic waves can occur in the later stages of evolution of plasma instability. The fourth field interaction is an important mechanism [4].

For the cubic–quintic NSE (1), on the other hand, some theoretical work has been obtained analytically and numerically. Puskharov *et al* [5] obtained solitary wave solutions. Cowan *et al* [6] showed numerically these are not solitons, but behave like quasi-solitons. Gagnon *et al* [7, 8] presented a large set of exactly analytic travelling-wave solutions for such a model. Zhou *et al* [9] also show that a special periodic solution can be reduced to a solitary wave as pseudo-energy being zero. The effects of the quintic nonlinear term enhance the amplitude and width of the solitary wave as compared with that for the cubic Langmuir soliton. Cloot *et al* [10] qualitatively proved the existence of bounded solutions by use of the invariants for quasi-particle number and energy. Their numerical results displayed the well known Fermi–Pasta–Ulam (FPU) recurrent phenomenon. The recurrence is only a special feature for such a system. In high space dimensions, the Lie symmetry group was analysed by Gagnon and Winternitz [11, 12]. They obtained a class of group-invariant solutions. Lemesnrier *et al* [13] pointed out that the focusing singularities can occur in two- or three-dimensions. Also, the stable particle-like solutions may exist in this dynamic system [3]. Even in the case of one-dimension, however, (1) for $g \neq 0$ does not belong to the class of integrable nonlinear evolution equations [14], which means that it cannot be solved by the inverse-scattering method and solitons and multi-solitons are not to be expected [7]. Many new and very interesting dynamic properties of the system, such as the stochastic propagation of wavefields, can appear [9]. In this paper, we theoretically and numerically show that the stochasticity results from the irregular homoclinic orbit (HMO) crossings, and we report that the quintic Hamiltonian perturbation to the cubic NSE can lead to the spatiotemporal complexity of fields. The mechanism of these complicated dynamics is also analysed in terms of energy spectra.

2. Homoclinic orbit crossings

In the case of one-dimension, the integrability of the cubic NSE can be verified by finding the Lax pair. A class of periodic solutions and solitons can be obtained by inverse-scattering transform. The solitons developed by modulational instabilities to a homogeneous solution keep their spatially coherent structures and temporally periodic evolutions. In order to analyse the dynamic properties of the cubic–quintic NSE (1), we consider a homogenous solution being

$$E_s(t) = E_0\, e^{it} \tag{2}$$

where E_0 satisfies with

$$E_0 = 0 \pm \sqrt{\frac{1 \pm \sqrt{1-4g}}{2g}}\,. \tag{3}$$

If we represent the solution of (1) in the form

$$E(X, t) = E_s + \delta E(X, t) \tag{4}$$

and linearize around E_s, we have

$$\begin{bmatrix} i\partial_t + \partial_{xx} + L(|E_0|^2) & h(t) \\ h^*(t) & -i\partial_t + \partial_{xx} + L(|E_0|^2) \end{bmatrix} \begin{bmatrix} \delta E \\ \delta E^* \end{bmatrix} = 0 \tag{5}$$

where L and h are expressed as following form

$$\begin{aligned} L(|E_0|^2) &= 2|E_0|^2 - 3g|E_0|^2 \\ h(t) &= E_s^2(t)(1 - 2g|E_0|^2). \end{aligned} \tag{6}$$

In the further analyses, the unstable wavenumber and the linear growth rate must be defined. Considering the homogeneous solution $E_s(t)$ to be modulated by the very small perturbation, we also represent the solution of (1) in the form [17]

$$E(X, t) = E_0\, e^{it} + \delta E_+\, e^{i(KX+\Omega t)} + \delta E_-\, e^{-i(KX+\Omega^* t)} \tag{7}$$

where δE_+ and δE_- are all much smaller than E_0. According to the linearizing analysis, we can obtain the linear disperse relation as follows

$$\Omega^2 = -K^2[2E_0^2(1-2gE_0^2) - K^2]. \tag{8}$$

When $0<|K|<K^* = E_0[2(1-2gE_0^2)]^{1/2}$, Ω is an imaginary number and is defined as $\Omega = \mathrm{i}\gamma$, where γ is called as the linear growth rate of modulational instability. Thus, the linear growth rate as a function of an unstable wavenumber K is

$$\gamma(K) = K[2E_0^2(1-2gE_0^2) - K^2]^{1/2} \tag{9}$$

for $0<|K|<K^* = E_0[2(1-2gE_0^2)]^{1/2}$. If we consider the periodic boundary conditions, the eigenfunction of (5) can be chosen as

$$\begin{bmatrix} \delta E \\ \delta E^* \end{bmatrix} = \begin{bmatrix} \varepsilon \mathrm{e}^{\mathrm{i}\lambda t} \\ \varepsilon^* \mathrm{e}^{-\mathrm{i}\lambda^* t} \end{bmatrix} \cos(KX) \tag{10}$$

where ε, ε^* are the small parameters.

It is convenient for the following numerical experiments that we take $K = K_{\max} = E_0(1-2gE_0^2)^{1/2}$, which corresponds to the maximum instability mode.

Hence, we easily get

$$\lambda = 1 \pm \mathrm{i}\frac{-1+4g+\sqrt{1-4g}}{2g} \tag{11}$$

for $E_0 = \pm\{[1-(1-4g)^{1/2}]/2g\}^{1/2}$ with $0 \leqslant g < \frac{1}{4}$. Thus, the amplitude of the eigenfunction of the linearizing operator can be written as

$$|\delta E(X,t)| = c_1 \cos(K_{\max}X)\,\mathrm{e}^{At} + c_2 \cos(K_{\max}X)\,\mathrm{e}^{-At} \tag{12}$$

where c_1 and c_2 are the small real parameters, and $A = [-1+4g+(1-4g)^{1/2}]/2g$. If we construct phase-space $(|E(X,t)|, |E(X,t)|_t)$, obviously, $E_0 = \pm\{[1-(1-4g)^{1/2}]/2g\}^{1/2}$ with $0 \leqslant g < \frac{1}{4}$ correspond to the hyperbolic fixed points. It is shown that $E_0 = \pm\{[1+(1-4g)^{1/2}]/2g\}^{1/2}$ and $E_0 = 0$ correspond to the elliptic points by means of the same process.

In the numerical experiment, on the other hand, we choose the initial condition as

$$E(X,0) = E_0 + \varepsilon\mathrm{e}^{\mathrm{i}\theta}\cos(K_{\max}X) \tag{13}$$

here ε is the small real parameter, $E_0 = \{[1-(1-4g)^{1/2}]/2g\}^{1/2}$, and (13) is called as simply initial condition for the cubic NSE [15]. Considering (12) and (13), we have that the unstable manifolds for possessing the saddle point $(|E_0|, 0)$ correspond to $\theta = 45°$ and $\theta = 225°$, and the stable manifolds correspond to $\theta = 135°$ and $\theta = 315°$, as sketched in figure 1. In the numerical processes, the periodic length of system is taken as $L = 2\pi/K_{\max}$, and the splitting-time-step spectral method [16] has been improved in order to increase the time accuracy. The available modes for fast Fourier transform and the time-step are considered, which depend on the accuracy of the conserved quantities being preserved to 10^{-8}.

In order to analyse the dynamic properties of the cubic–quintic NSE (1), first, we simply discuss the cubic NSE. For $g = 0$, the integrability can be illustrated in terms of the exactly periodic recurrent motion in phase-space. Moon [17] experimentally guessed the fixed point $(1,0)$ could be a saddle point, which has been confirmed in the above linearized analysis. In addition, the orbit to possess the hyperbolic fixed point $(1,0)$ should correspond to the homoclinic one. As far as a finite dimensional dynamic system is concerned, the stable and unstable orbits for the hyperbolic fixed point would smoothly be joined to each other if the unperturbed system is taken to be integrable. For a Hamiltonian perturbation the orbits generically intersect transversely, leading

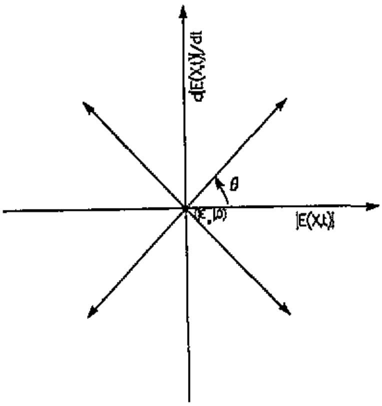


Figure 1. Local phase flows of saddle point ($|E_0|$, 0), which are obtained in terms of (12).

to an infinite number of homoclinic points and chaotic motion [18]. For our continuum Hamiltonian system, the integrability of the cubic NSE is also illustrated in terms of such an idea. Making use of the results obtained in the linearized stability analysis, we choose the initial parametric values as $g=0$, $E_0=1$ and $\theta=45°$. From figure 2(a), we observe that the stable manifold $W^{(s)}$ smoothly joins with the unstable manifold $W^{(u)}$. The orbit to possess the saddle point (1, 0) corresponds to the homoclinic one. However, when system is added in the quintic Hamiltonian perturbation ($g\neq 0$), we find that stable and unstable manifolds in phase-space do not smoothly join together (see figure 2(b)), which illustrates that the current system is near integrable. On the other hand, we note that there are not an infinite number of homoclinic points in figure 2(b). In fact, our phase-space is only the projection of a high-dimensional space, where the information of phase for wave fields is not contained. Therefore, there may exist some difference between our HMO crossings with those in a finite-dimensional Hamiltonian system. To further analyse the chaotic behaviour, we must discuss the long-time evolution of wave fields.

3. Spatiotemporal complicated dynamics

In section 2, we have shown that the irregular HMO crossings would appear when the quintic Hamiltonian perturbation is considered. To analyse the long-time behaviour, we also deal with cases of $g=0$ and $g\neq 0$.

For the cubic NSE, the integrability can be displayed from figure 3. Figure 3(a) indicates the amplitude of fields is periodically temporal evolution. The periodic behaviour can also be illustrated by the plot of power spectrum (figure 3(c)), where the foundational frequency is about $\omega_0=0.549$. The exactly periodic recurrent solution is exhibited in figure 3(b, d). These phenomena account for the integrability of the cubic NSE.

As $g\neq 0$, however, a completely different dynamic behaviour is shown in figure 4. Taking into account the quintic Hamiltonian perturbation, we find that the periodic behaviour disappears. The temporal evolution for the amplitude of fields in figure 4(a) indicates the typically chaotic characteristic. As shown in figure 4(b), the irregular HMO crossings are very clear. A continuous non-periodic spectrum which obviously accounts for the chaotic characteristic is described in figure 4(c). These behaviours

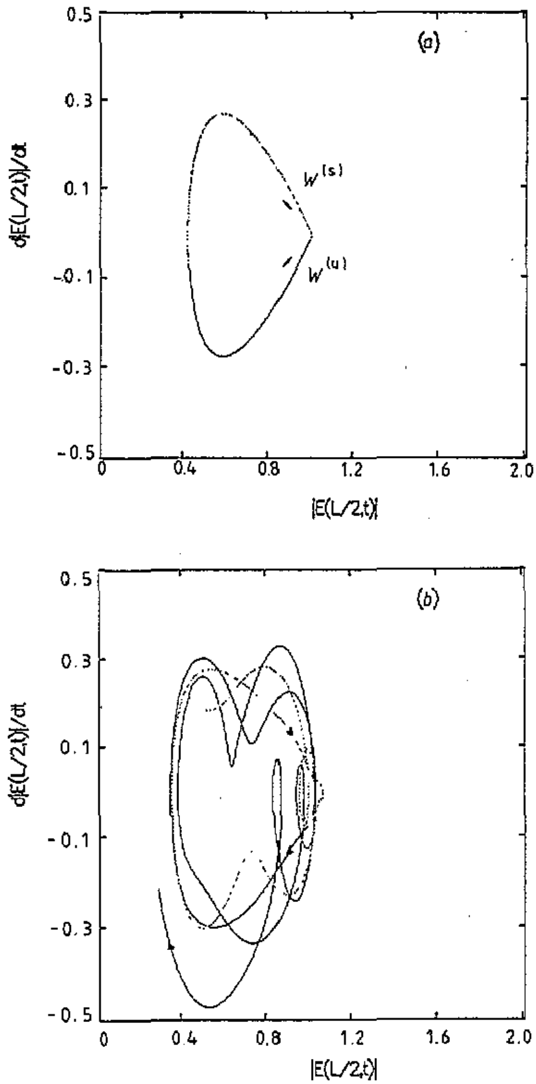


Figure 2. Stable ($W^{(s)}$) and unstable ($W^{(u)}$) manifolds for the hyperbolic fixed point $(|E_0|, 0)$, where solid line is computed with $t>0$, and dotted line is computed with $t<0$. (a) Integrable cubic NSE; the orbits smoothly join. (b) Non-integrable cubic–quintic NSE; the stable and unstable orbits intersect.

show that the quintic Hamiltonian perturbation leads to chaos. It is noted from figures 3 and 4 that the quintic nonlinear term in (1) behaves not completely like a general dissipative perturbation. Some more complicated dynamic phenomena cannot be explained only in terms of the non-integrable perturbation, such as the 'foundational' frequency (corresponding to the first maximum peak) in figure 4(c) is about $\omega_0 = 0.84$. But such a complicated behaviour will not be discussed in this paper. From figure 4(d), in addition, we observe that the spatially localized structures are still kept in the propagative processes of fields, but are considerably irregular. The presence of these irregular spatial patterns shows the recurrence broken down. Thus, the spatiotemporal complicated behaviour is realized in the present dynamic system.

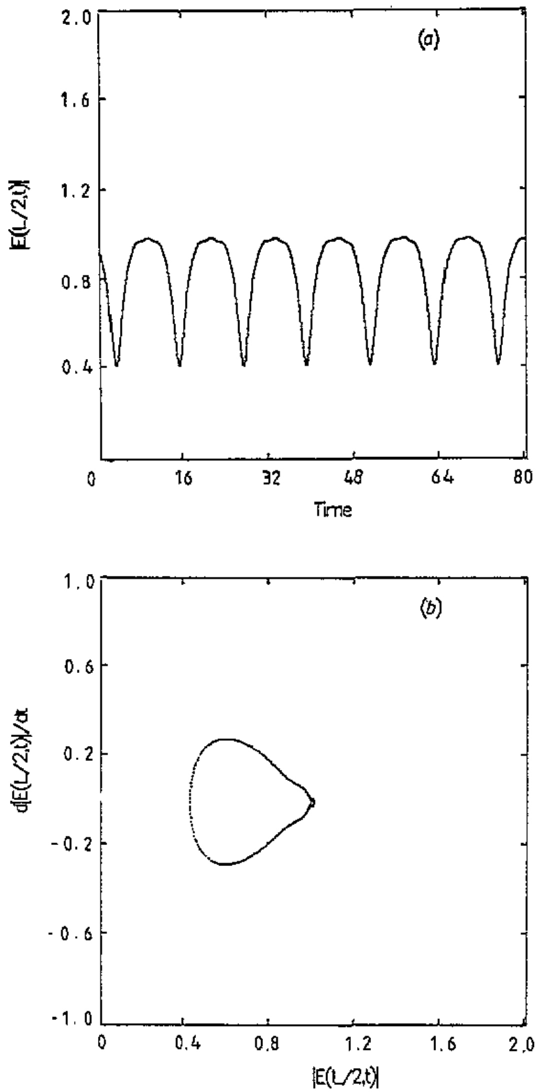


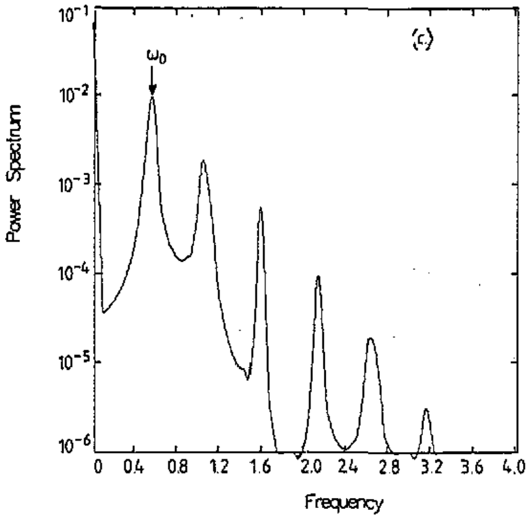
Figure 3. Solution of the cubic NSE with $\theta=45°$ and $\varepsilon=0.1$: (a) the periodic evolution for the amplitude of fields; (b) the phase-space orbit; (c) the power spectrum, where the foundational frequency $\omega_0\approx 0.549$; and (d) contours of $|E(X,t)|^2=$constant, where $t=0\sim 100$.

To analyse the mechanism that leads to spatiotemporal complexity, we further investigate the processes that the energy in Fourier modes evolves with time. For (1), we define energy of system as

$$H=\int |E|^2\,\mathrm{d}X. \tag{14}$$

In Fourier space, it can be rewritten as

$$H=\sum_K H_K=\sum_K |E_K|^2. \tag{15}$$



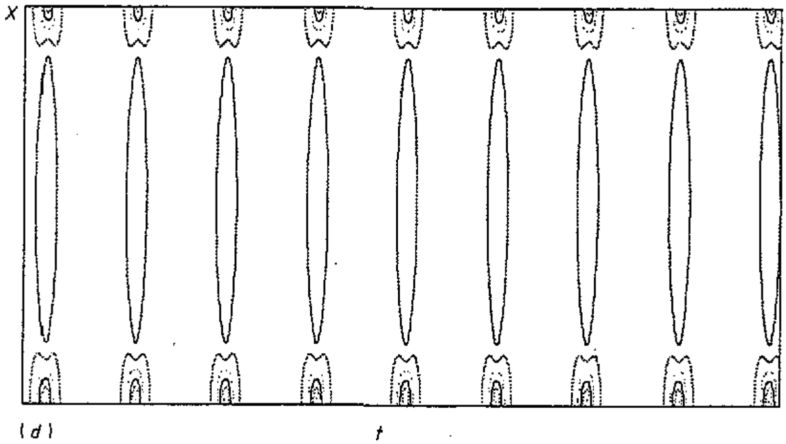


Figure 3. (continued)

Here K stands for the kth Fourier mode. The time evolutions of the amplitude for fields in Fourier space, which also correspond to the evolutions of energy in Fourier modes, i.e., $H_K = |E_K|^2$, are shown in figure 5. Obviously, the large part of energy in system is lain in the first and the second modes. For the case of $g = 0$, the time evolutions of energy in all modes is periodic, which is consistent with periodic recurrent solution (see figure 3). The amplitude of soliton structures is dominated by the total energy of system, the width and patterns of coherent structures are associated with energy partition in the high modes. Therefore, the evolution of fields is temporally periodic and spatially coherent. For the case of $g \neq 0$, but, figure 5 shows that energy in the system, which is initially confined to the lowest mode, would spread to many higher modes due to the nonlinear instability, but would not regroup into the original lowest mode. Because the initial energy is added to the maximum modulational instability mode, the main energy would be controlled by the finite modes in the processes of evolutions. We see from figure 5 that energy contained in the first two modes dominates the spatially localized structures being kept. The energy contained in higher modes,

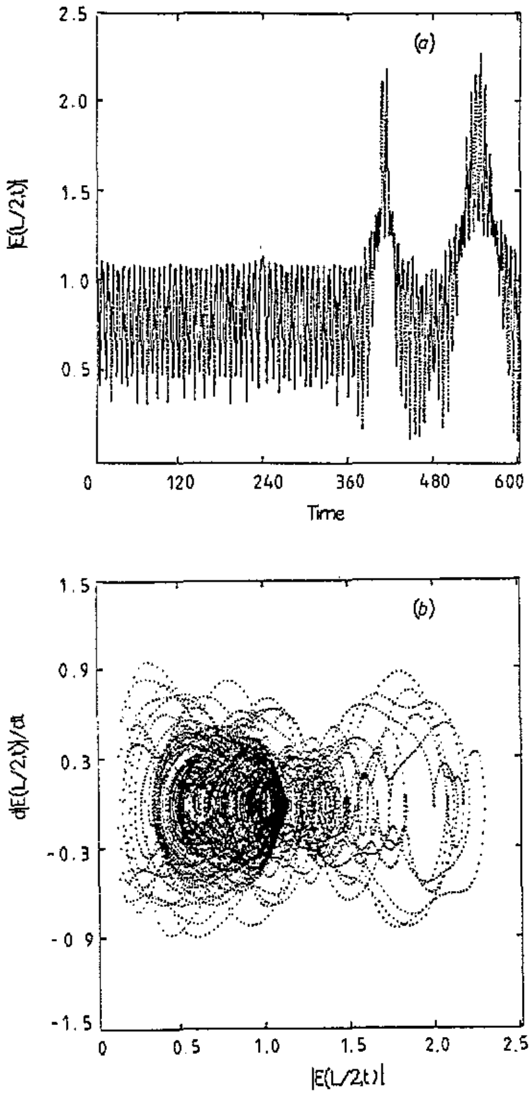


Figure 4. Solution of the cubic-quintic NSE with $g=0.1$, $\theta=45°$ and $\varepsilon=0.1$: (a) chaotic evolution for the amplitude of fields; (b) the phase-space trajectory illustrates the irregular HMO crossings; (c) a continuous noise-like power spectrum; and (d) contours of $|E(X,t)|^2=\text{constant}$, where $t=550\sim600$.

for example the clearly stochastic behaviours in the third and the fourth modes, leads to the presence of various spatial patterns of the localized structures. The spatio-temporal complexity as shown in figure 4(d) would be associated with the stochastic evolutions of energy lain in all modes.

4. Conclusions and discussions

The essentially important characteristic of the cubic-quintic NSE (1) has been discussed theoretically and numerically. We show that the quintic Hamiltonian perturbation may

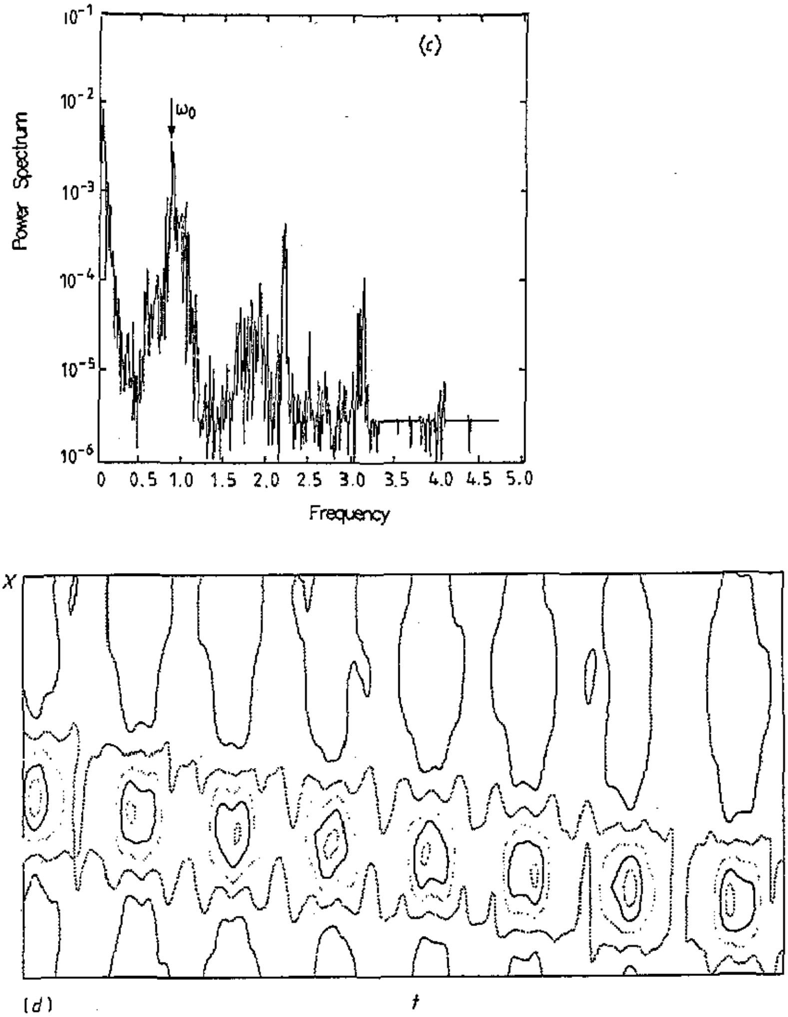


Figure 4. (continued)

lead to the integrability of system broken down. The propagation of fields is temporally stochastic but keeps their spatially localized patterns that developed by modulational instability to a homogeneous solution. The spatiotemporal complexity is due to the irregular HMO crossings. We also show that these complicated dynamic behaviours are associated with the fact that energy contained in Fourier modes stochastically evolves with time. In particular, the stochasticity of energy in higher modes would lead to the presence of various irregularly spatial structures.

Our current results are of significance in physics. In plasma turbulence, for example, the cubic NSE describes the nonlinear interaction between Langmuir wave and ion-acoustic wave under the subsonic regime, where the second-order fields are considered. In the evolutive initial stage, some physical phenomena can well be explained in terms of the cubic NSE. In the nonlinear strong turbulent stage, however, the cubic NSE may be no longer valid. The other physical effects, such as damping and dissipation, etc, would lead to the presence of the complicated dynamics of Langmuir fields. On the other hand, the high order field interaction has to be considered in the evolutive later

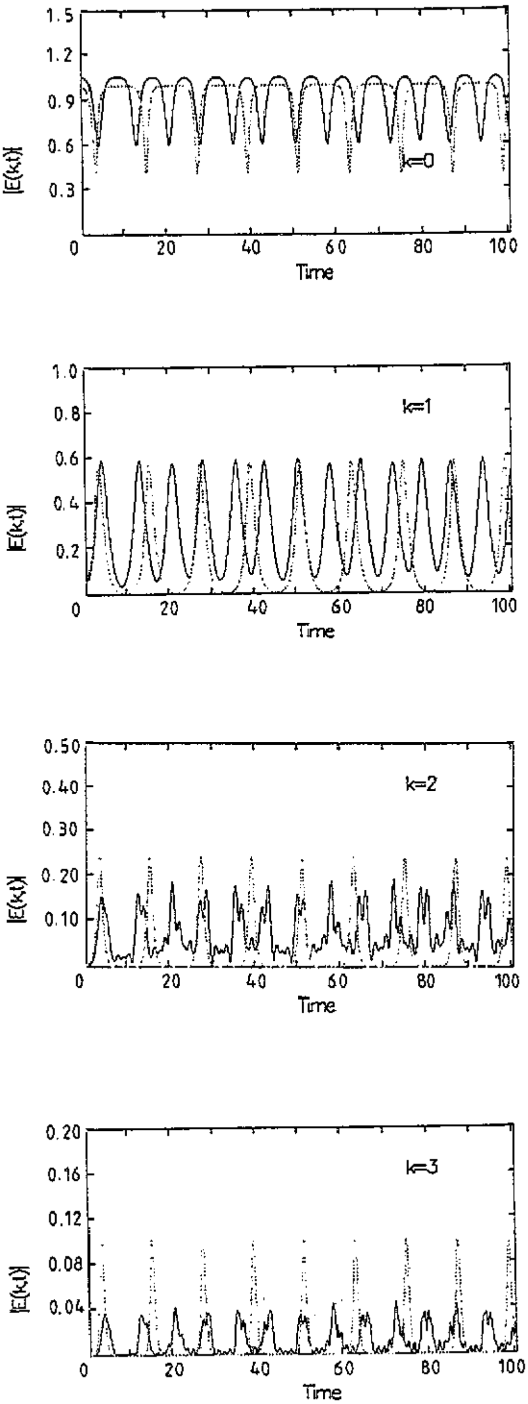


Figure 5. The time evolution of the amplitude for fields in Fourier space (also corresponding to the energy in Fourier modes, i.e., $H_K = |E(K)|^2$) for (1) with $\theta = 45°$, $\varepsilon = 0.1$, where dotted curves correspond to solutions of the cubic NSE and solid curves correspond to solutions of cubic–quintic NSE with $g = 0.1$, respectively.

stage of plasma instability with the increase of the amplitude of fields. Although the dynamic model (1) only involves in the fourth-order field interaction, it is shown that the quintic Hamiltonian corrections may also drive the chaotic evolutions of Langmuir fields. In addition, our research would also be interesting in nonlinear optics, etc.

Finally, we should mention the main difference between our work and those discussed by some authors. For the study of the spatiotemporal complexity in Langmuir turbulence, Doolen *et al* [19] first investigated the driven dissipative Zakharov equations numerically. Their studies demonstrated the co-existence of coherent spatial structures—cavitons—and temporal chaos. After their work, a set of dynamic models have been discussed in studying the process of coherence to turbulence. But, much work deals with the driven damping, force dissipation and nonlinear inhomogeneous media [20]. A rich variety of spatiotemporal complexity, which suggests that a low-dimensional chaotic attactor exists in an infinite-dimensional system, has been observed [21]. For the continuum Hamiltonian system, however, the problem becomes extremely difficult for general initial conditions since the system does not reduce to a finite-dimensional system as that in the case of dissipative perturbations though the existence of the solitary waves could slow down the dimensions of system. Therefore, more care is needed to investigate chaos of the continuum Hamiltonian system.

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